SOME RELATIONS BETWEEN POLYGONAL FIGURATIVE NUMBERS

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Abstract: *Number as a basic element of human knowledge is an important subject of study in various scientific disciplines. From the ancient period to the present day, many great scientists have searched for the basic law according to which nature rules, but also other components of human life. In the 6th century BC Pythagoras claimed that the world is ruled by numbers. The basic Pythagorean dogma was: "Everything is a number". Figurative numbers, which were studied by the Pythagoreans, as well as many scientists after them, occupy a special place in the world of numbers. In this paper we will be dealing with combined relations between polygonal figurative numbers. Some effective results for relations between figurate numbers and natural numbers, are proven. Starting with the proven assumption that every natural number is a sum of the square and two triangular numbers, we have come to interesting relations between m-angled polygonal numbers, squares and triangular numbers.*

Key words*: squares, triangular numbers, mixed sums, representations of natural numbers.*

1. INTRODUCTION

The figurative method of displaying numbers is being applied since the ancient times (see [1] and [2]). Over the time, the initial Pythagorean triangular and square numbers were supplemented by a whole class of polygonal, polyhedral and other figurative numbers. In the past centuries, many famous mathematicians dealt with them and proved a lot of interesting links among them (see [3]). Contemporary researchers have dealt with relations between natural, squared and triangular numbers (see [4]-[13]). Also, triangular and squared numbers (see [14]) as well as polygonal and pyramidal numbers (see [15]). Generator functions for the figurative numbers of regular polyhedrons were also investigated (see [16]).
The author of this paper had conducted a thorough research of the usefulness of introducing figurate numbers in the regular mathematics lessons in primary and secondary schools (see [17]).
After completing the before mentioned research, which was conducted among the students of the first grade of high school, we came to the next conclusion: Working with figurative numbers contributes to the development of the ability to notice laws in relations among the numbers and to apply those observed laws in solving tasks with sequences and sets of numbers. The results obtained in the research provide an incentive for further research into the relations between the figurative numbers and the potential introduction of these laws into regular mathematics education.
In this paper we investigate the mixed sums of natural and polygonal numbers. There are many proven results in this field that were obtained in the past. The results that contemporary researchers have come up with in these modern times open the door to new research and new results.
Motivated by Sun's theorem (see [11]), in which is proven that each natural number is the sum of the squared and two triangular numbers, the authors of this paper were looking for the connection between an arbitrary m-angular polygonal number, a squared and a triangular numbers. The final results are shown in the Theorem 2.4. section Main results.

2. METHODOLOGICAL FRAMEWORK FOR RESEARCH

2.1. Background

Figurative numbers can be represented by a discrete geometric pattern with equally spaced points, with each point representing a unit (Figure 1).

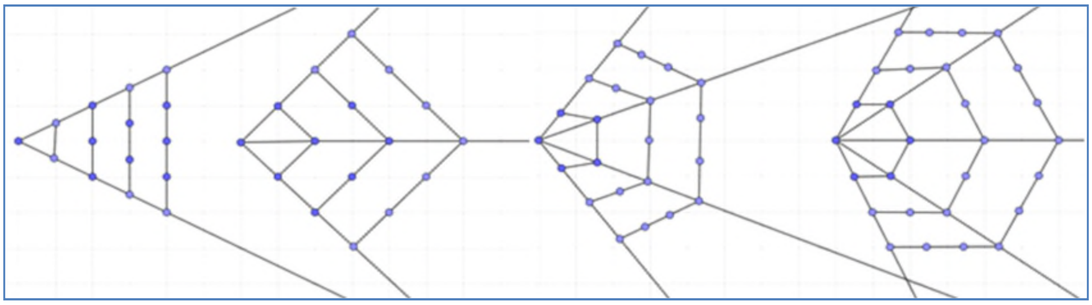

Figure 1: Triangular, square, pentagonal and hexagonal numbers

They can also be represented by an algebraic formula by which all numbers of this array are generated. We denote by $S_3(n)$, $S_4(n)$, $S_5(n)$, $S_6(n)$, the **n**th member of a series of triangular, square, pentagonal and hexagonal numbers. Then

$$S_3(n) = \frac{n(n+1)}{2}; \quad S_4(n) = n^2; \quad S_5(n) = \frac{n(3n-1)}{2}; \quad S_6(n) = n(2n-1). \tag{1}$$

The study of the relationships between polygonal numbers has a long history. Nicomachus of Alexandria (1nd century BC) came to the following conclusion:

$S_m(n) = S_{m-1}(n) + S_3(n-1)$, $m>3$, m, n ϵ N (see [3, chapter 1, p. 20]).

Remark 1: $S_m(0) = 0$ by definition.

Theon of Smyrna (2nd century AC) showed that

$S_3(n) + S_3(n-1) = S_4(n)$, n ϵ N, (see [3, chapter 1, p. 12]).

Bachet de Mėziriac (17th century AC) showed that

$S_m(n) = S_3(n) + (m-3) S_3(n-1)$, $m \geq 3$, m, n ϵ N (see [3, chapter 1, p. 21]).

Pierre Ferme (17th century AC) claimed:

"Every natural number can be written as the sum of three triangular numbers" (see [12, p.2]).

2.2. Auxiliary results

Let us prove Nicomachus claim that we express in Lemma 1.

Lemma 1:

$$S_6(n) - S_5(n) = S_5(n) - S_4(n) = S_4(n) - S_3(n) = S_3(n-1), \ n \in N.$$

Proof: Using the formulas mentioned in (1) we have:

$$S_6(n) - S_5(n) = n(2n-1) - \frac{n(3n-1)}{2} = \frac{n(4n-2-3n+1)}{2} = \frac{n(n-1)}{2} = S_3(n-1) \tag{2}$$

$$S_5(n) - S_4(n) = \frac{n(3n-1)}{2} - n^2 = \frac{n(3n-1)-2n^2}{2} = \frac{n(3n-1-2n)}{2} = \frac{n(n-1)}{2} = S_3(n-1) \tag{3}$$

$$S_4(n) - S_3(n) = n^2 - \frac{n(n+1)}{2} = \frac{2n^2 - n(n+1)}{2} = \frac{n(2n-n-1)}{2} = \frac{n(n-1)}{2} = S_3(n-1) \tag{4}$$

Corollary 1: For n, m ϵ N ∧ m ≥ 3

$S_4(n) = S_3(n) + S_3(n-1)$,

$S_5(n) = S_3(n) + 2S_3(n-1)$,

$S_6(n) = S_3(n) + 3S_3(n-1)$,

$S_m(n) = S_3(n) + (m-3)S_3(n-1)$.

Proof:

Based on (4) we have $S_4(n) - S_3(n) = S_3(n-1) \Rightarrow S_4(n) = S_3(n) + S_3(n-1)$ (5)

Based on (3) we have $S_5(n) - S_4(n) = S_3(n-1) \Rightarrow S_5(n) = S_4(n) + S_3(n-1)$

and based on (5) $\Rightarrow S_5(n) = S_3(n) + 2S_3(n-1)$ (6)

Based on (2) we have $S_6(n) - S_5(n) = S_3(n-1) \Rightarrow S_6(n) = S_5(n) + S_3(n-1)$

and based on (6) $\Rightarrow S_6(n) = S_3(n) + 3S_3(n-1)$ (7)

Bachet de Mẻziriac (see [3, chapter 1, p. 21]) proved that

$$S_m(n) = S_3(n) + (m-3)\, S_3(n-1), \quad m \geq 3,\ m, n \in N \tag{8}$$

Theorem 2.1. For every $n \in N$ and $n > 1$ the next equalities are valid

(i) $S_4(n) = n + 2\cdot\sum_{i=1}^{n-1} i$

(ii) $S_5(n) = n + 3\cdot\sum_{i=1}^{n-1} i$

(iii) $S_6(n) = n + 4\cdot\sum_{i=1}^{n-1} i$

Proof:

For $n \in N$ i $n > 1$

(i) $n + 2\cdot\sum_{i=1}^{n-1} i = n + 2\cdot(1 + 2 + 3 + \ldots + n - 1) =$

$= 1 + 2 + 3 + \ldots + n-1 + n + 1 + 2 + 3 + \ldots + n - 1 = \frac{n(n+1)}{2} + \frac{(n-1)n}{2}$

$= S_3(n) + S_3(n-1) = S_4(n)$ based on (5).

(ii) $n + 3\cdot\sum_{i=1}^{n-1} i = n + 3\cdot(1 + 2 + 3 + \ldots + n - 1) =$

$= 1 + 2 + 3 + \ldots + n-1 + n + 2\cdot(1 + 2 + 3 + \ldots + n - 1) =$

$= \frac{n(n+1)}{2} + 2\cdot\frac{(n-1)n}{2} = S_3(n) + 2\cdot S_3(n-1) = S_5(n)$ based on (6).

(iii) $n + 4\cdot\sum_{i=1}^{n-1} i = n + 4\cdot(1 + 2 + 3 + \ldots + n - 1) =$

$= 1 + 2 + 3 + \ldots + n-1 + n + 3\cdot(1 + 2 + 3 + \ldots + n - 1) =$

$= \frac{n(n+1)}{2} + 3\cdot\frac{(n-1)n}{2} = S_3(n) + 3\cdot S_3(n-1) = S_6(n)$ based on (7).

Theorem 2.2. For $m, n \in N$, $m \geq 3$, $n > 1$

$S_m(n) = n + (m-2)\sum_{i=1}^{n-1} i$

Proof:-

If $m \geq 3$, $n > 1 \wedge m, n \in N$

$n + (m-2)\sum_{i=1}^{n-1} i = n + (m-2)\cdot(1 + 2 + 3 + \ldots + n-1) =$

$= 1 + 2 + 3 + \ldots + n-1 + n + (m-3)\cdot(1 + 2 + 3 + \ldots + n - 1) =$

$= \frac{n(n+1)}{2} + (m-3)\cdot\frac{(n-1)n}{2} =$

$= S_3(n) + (m-3)\cdot S_3(n-1) = S_m(n)$ based on (8).

Theorem 2.3. [11, Theorem 1]

Any $n \in N$ is a sum of an even square and two triangular numbers. Moreover, if $\frac{n}{2}$ is not a triangular number then

$$|\{(x,y,z) \in Z \times N \times N: x^2 + t_y + t_z = n \ \wedge 2 \nmid x\}|$$
$$= |\{(x,y,z) \in Z \times N \times N: x^2 + t_y + t_z = n \ \wedge 2 \mid x\}|$$

In the theorem is claimed that every natural number is a sum of an even square and two triangular numbers. In addition, if $n \in N$ and $n \neq 2t_m$ for each $m \in N$ than n is also both sum of one odd square and two triangular numbers.

Remark 2: t_m is label for m-th triangular number.

The proof of this theorem is given in [11].

Remark 3: In before mentioned paper [11, Theorem 3] Sun also proves the claim that if a, b, c are positive integers with $b \geq c$, then each $n \in N$ can be written in the form $ax^2 + bt_y + ct_z$ with $x,z,y \in Z$ where (a, b, c) are from the following set of vectors:
{(1,1,1), (1,2,1), (1,2,2), (1,3,1), (1,4,1), (1,4,2), (1,5,2), (1,6,1), (1,8,1), (2,1,1), (2,2,1), (2,4,1), (3,2,1),(4,1,1), (4,2,1)}.

2.3. Main results

The main results of our work are presented in the next theorem.
Theorem 2.4. For $m, n, x, y, z \in N \wedge m \geq 3$

$$S_m(n) = x^2 + S_3(y) + S_3(z) + (m-2)\, S_3(n-1)$$

where $k \in Z$ and

$$x = \begin{Bmatrix} 2k, & \text{if } n=2S_3(m) \\ 2k-1 & \text{if } n\neq 2S_3(m) \end{Bmatrix}.$$

Proof:

According to proven Theorem 2.2. we have

$$S_m(n) = n + (m-2)\sum_{i=1}^{n-1} i \text{ , for } m, n \in N,\ m \geq 3, n > 1.$$

On the other hand

$$\sum_{i=1}^{n-1} i = 1 + 2 + 3 + \ldots + n - 1 = \frac{(n-1)n}{2} = S_3(n-1) \tag{9}$$

According to Theorem 2.3., which confirms that each natural number can be written as the sum of the even square and two triangular numbers, we can conclude that every natural number can be written as follows:

$$n = x^2 + S_3(y) + S_3(z),\ x, y, z \in N \ \wedge\ x = 2k, k \in Z. \tag{10}$$

On the basis of (9), (10) and proven Theorem 2.2. it follows that

$$S_m(n) = x^2 + S_3(y) + S_3(z) + (m-2)\, S_3(n-1),\ m \geq 3,\ m, n, x, y, z \in N \ \wedge\ x = 2k, k \in Z$$

by which Theorem 2.4. is proved for $n \in N$. If the assertion is valid for $n \in N$ then also holds for $n = 2S_3(m)$ because $2S_3(m) \in N$. So Theorem 2.4. is proved for $n = 2S_3(m)$.

Similarly, according to Theorem 2.3. and previously exposed, we have the following:

if $n \neq 2S_3(m)$ for each $m,n \in N$ and $m \geq 3$ than n is also both sum of one odd square and two triangular numbers, i.e.

$$n = x^2 + S_3(y) + S_3(z),\ x, y, z \in N \ \wedge\ x = 2k-1, k \in Z, \tag{11}$$

so, on the basis of (9), (11) and proven Theorem 2.2. it follows that
$S_m(n) = x^2 + S_3(y) + S_3(z) + (m-2)\, S_3(n-1),\ m \geq 3,\ m, n, x, y, z \in N$ i $x = 2k-1, k \in Z$

by which Theorem 2.4. is proved for $n \neq 2S_3(m)$. That's why the evidence is finished.

3. CONCLUSION

In this paper we showed the connection between an arbitrary m-angular polygonal number, a squared and a triangular numbers. The result we have come is another mix sums of natural and polygonal numbers, and on that way it contributes to the theory of numbers. Legalities among the figurative numbers proved in Theorem 2.4. allow the generation of an arbitrary m-angular polygonal number using a mathematical apparatus as well as using a computer. That make them additionally interesting and useful from the standpoint of teaching mathematics and informatics.
The claim stated in Note 3, as well as other proven claims about mixed sums of figurate numbers, open the door to new research and new results in the field of figurate numbers.

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